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LETTER TO THE EDITOR

The Lie derivative of the vorticity vector in an isometric flow

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Abstract. It is shown that in an isometric flow the vorticity vector is invariant under the group of motions.

A fluid flow in general relativity is said to be isometric if there exists a time-like Killing vector ξ_α colinear with the fluid velocity vector u_α, ie if

$$u_\alpha = \xi_\alpha/\xi, \qquad \xi^2 = \xi_\alpha\xi^\alpha > 0, \qquad \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0. \tag{1}$$

Ciubotariu (1972a, b) has considered such flows in detail and has shown that

$$\mathscr{L}(\xi)\omega_{\alpha\beta} = 0 \tag{2}$$

$$h_\alpha\mathscr{L}(u)\omega^\alpha = 0 \tag{3}$$

where $\mathscr{L}(\xi)$ and $\mathscr{L}(u)$ denote the Lie derivatives with respect to ξ_α and u_α respectively, $\omega_{\alpha\beta}$ and ω^α are the vorticity tensor and the vorticity vector and h_α is the magnetic field four-vector. Result (3) is important as it can be regarded as a generalization of Ferraro's (1937) theorem of isorotation. It is the purpose of this note to show that besides (2) and (3) we also have in an isometric flow

$$\mathscr{L}(\xi)\omega^\alpha = 0. \tag{4}$$

Instead of starting from first principles, we make use of (2). The vorticity vector ω^α can be expressed in terms of $\omega_{\alpha\beta}$ as

$$\omega^\alpha = \tfrac{1}{2}\eta^{\alpha\beta\lambda\mu}\omega_{\beta\lambda}u_\mu \tag{5}$$

where $\eta^{\alpha\beta\lambda\mu}$ is the permutation tensor. As

$$\eta_{\alpha\sigma\rho\nu}\eta^{\alpha\beta\lambda\mu} = -\delta^\beta_\sigma(\delta^\lambda_\rho\,\delta^\mu_\nu - \delta^\mu_\rho\,\delta^\lambda_\nu) - \delta^\lambda_\sigma(\delta^\mu_\rho\,\delta^\beta_\nu - \delta^\beta_\rho\,\delta^\mu_\nu) - \delta^\mu_\sigma(\delta^\beta_\rho\,\delta^\lambda_\nu - \delta^\lambda_\rho\,\delta^\beta_\nu) \tag{6}$$

we have

$$\eta_{\sigma\rho\nu\alpha}\omega^\alpha = \omega_{\sigma\rho}u_\nu + \omega_{\rho\nu}u_\sigma + \omega_{\nu\sigma}u_\rho. \tag{7}$$

Now as ξ_α is a Killing vector,

$$\mathscr{L}(\xi)u_\alpha = 0. \tag{8}$$

Thus on taking the Lie derivative with respect to ξ_α of both sides of (7), we have

$$\eta_{\sigma\rho\nu\alpha}\mathscr{L}(\xi)\omega^\alpha = -\omega^\alpha\mathscr{L}(\xi)\eta_{\sigma\rho\nu\alpha}. \tag{9}$$

If we write $\mathscr{L}(\xi)\eta_{\sigma\rho\nu\alpha}$ out explicitly and operate on both sides of (9) by $\eta^{\sigma\rho\nu\beta}$ and note that

$$\eta^{\sigma\rho\nu\beta}\eta_{\sigma\rho\nu\alpha} = -6\delta^{\beta}_{\alpha} \tag{10}$$

and, for example, that

$$\eta^{\sigma\rho\nu\beta}\eta_{\sigma\rho\tau\alpha} = -2(\delta^{\nu}_{\tau}\delta^{\beta}_{\alpha} - \delta^{\beta}_{\tau}\,\delta^{\nu}_{\alpha}) \tag{11}$$

then we obtain

$$\mathscr{L}(\xi)\omega^{\beta} = -\omega^{\beta}\xi^{\tau}{}_{;\tau}. \tag{12}$$

But as ξ_{α} is a Killing vector, $\xi^{\tau}{}_{;\tau} = 0$, so that

$$\mathscr{L}(\xi)\omega^{\beta} = 0. \tag{13}$$

This is the same as saying that in an isometric flow, ω^{β} is invariant under the group of motions.

References

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